Mathematical Techniques in Finance
Tools for Incomplete Markets

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Abstract

The purpose of this document is to summarize the main points of the book "Mathematical Techniques in Finance - Tools for Incomplete Markets ".

Context

Books on derivative pricing come in all shades and colors. Some books give a brief introduction of derivatives at a leisurely pace and then suddenly the content becomes very mathematical. There are some books that have theorems and lemmas all through. There are some books that talk about risk-neutral pricing giving very little intuition about the concept. In the gamut of books available, I think this book stands out for a couple of reasons. The fact that we all live in *incomplete markets* is addressed right from the beginning. This immediacy has the reader's attention right away. If the markets are incomplete, i.e there are more state variables than the instruments, how does one hedge an exposure ? Can there be a perfect hedge ? If not, how does one compare between two or more hedging options? These are very practical questions for an options trader. An option trader intuitively knows that a perfect option hedge that is taught in a grad school is an idealistic scenario that holds good under a ton of assumptions. Real world is messy. Pick up any book where Black Scholes is derived; in 9 out 10 books, you will see measure theory as a prerequisite to understanding the content. This book though, does not to have a math heavy prerequisites as most of the book can be read with linear algebra, elementary calculus and probability knowledge. So, in a way, this book can be read by a wider audience. Even though this book uses many numerical simulations, the author also believes that

> There are computations one can do with pen and paper that even the fastest computers cannot perform.

Even though the book has 13 chapters and organized logically, the author suggests that one can take several trails for first, second or third pass through the book. One can follow discrete finance trail by working through Chapters 1, 2, 5, 6; One can go over the continuous finance trail by working through Chapters 8, 9, 10, 11 and 12; One can go over risk management trail by reading through Chapters 3,4 and 13. If you are like me who likes to read stuff that combines intuitive arguments and math, you might want to go over the entire book cover to cover. In this document, I will try to summarize the main points of the entire book that has taken the author more than three years to write.

The simplest model for financial markets

The first thing that is striking about this chapter is the use of vector and matrices to describe various concepts in finance. I have not seen this kind of approach in any of the Derivative pricing books that I have read till date. At its core, this chapter is on one period finite state model of financial markets. This model is probably the first model that is taught at MBA level and masters level courses. However the approach taken in this book is different, i.e. its purpose is to show how why one period model pricing works in the abstract word and why a simple variant of it makes it impossible to price a simple derivative security uniquely. How does a finite state model depart from reality ? Well, firstly, with a small number of scenarios, it provides only a patchy coverage of the actual outcomes, and secondly we do now know the objective probabilities of each scenario. We only have our subjective opinion of how much they might be.

The chapter introduces the idea of *complete market* from a linear algebra perspective. If there are n states of randomness and there are m assets whose payoffs in the n states are captured by column vectors, then the whole information can be conveniently summarized as $A^{m\times n}$ matrix. Even though the terms in the chapter look fancy such as *basis assets*, *focus assets*, *redundant asset*, *price of an Arrow-Debreu security*, from a linear algebra perspective, they are all nothing but a spin on the standard terms such as basis of a subspace, dimension of a subspace, an elements in the subspace, inverse of a matrix etc. In simple terms, a market is said to be complete if there are enough independent assets in a n state world to replicate any payoff in any state.

If you wear linear algebra lens, you can breeze past through this chapter with ease. Terms such as Arrow-Debreu matrix, a collection of securities where the jth security pays off 1 in the j th state and 0 everywhere else, is nothing but an identity matrix. Hence the inverse of a payoff matrix A has individual columns that represent replicating portfolios to individual Arrow-Debreu securities. I think the author's motivation behind explaining via linear algebra vocabulary is that asset replication can be easy to grasp by making a statement such as "for the system of equations $Ax = b$, there is a unique solution". If the same thing has to be said using finance terminology in a continuous time word, one might have to introduce the concept of hedging, martingale pricing and whole lot of machinery to make it rigorous. By the end of this chapter, a reader will get a decent idea about the concept of replication and arbitrage pricing. The takeaway from this chapter is that a hedge for a focus asset is possible only when the market is complete and it is unique only if there are no redundant basis assets.

In reality though, one can never have enough assets to make the market complete. Hence any asset introduced the market to hedge a specific kind of risk is a welcome move. This explains why over the years products such VIX, VIX futures, VIX options have been introduced to increase the dimension of the marketed subspace of all existing securities.

Arbitrage and Pricing in the One-Period Model

This chapter starts with payoff of a portfolio of basis assets $A^{m\times n}$ and lists out the conditions when the focus asset can be replicated or not.. This basically entails solving $Ax = b$. Well, the A matrix can be skinny matrix $m > n$, fat matrix $m < n$ or a square matrix $m = n$. The solution to $Ax = b$ is a standard problem in elementary linear algebra. However when you mix in finance terminology, you get fancy statements such as :

- complete market + no redundant securities + focus asset is redundant, then one can replicate the focus asset in a unique way

- complete market + no redundant securities + focus asset is non-redundant, then this scenario is an impossible one
- incomplete market + no redundant securities + focus asset is redundant, then one can replicate the focus asset in a unique way via Generalized inverse of A
- incomplete market + no redundant securities + focus asset is non-redundant, then one cannot replicate the focus asset
- incomplete market + redundant securities + focus asset is redundant, there are many ways to replicate the focus asset
- incomplete market + redundant securities + focus asset is non-redundant, then one cannot replicate the focus asset

The chapter then introduces *type I* arbitrage and *type II* arbitrage. The latter type of arb implies that there is a mispriced redundant basis asset. The absence of second type of arbitrage means that every marketed payoff has a unique price(*law of one price*) and that prices are linear. With these concepts in place, the chapter goes on to state *Arbitrage theorem.*

> A market with n securities, m states of the world, a security payoff matrix $A \in \mathbb{R}^{m\times n}$ and a security price vector $S \in \mathbb{R}^n$ admits no arbitrage if and only if there is a strictly positive state price vector $\psi \in \mathbb{R}^m$ consistent with the security price vector S, i.e. $S = A^T\psi$.

The reason one needs to use the concept of state prices and no arbitrage conditions is to establish the following relation

$$S = \frac{1}{R_f} A^T q$$

The above equation means that price of each security is the present discounted value of its expected payoff under probabilities q_j, which are called *risk neutral probabilities.* This is a particularly nice way of understanding about risk neutral probabilities. The way to interpret q is that it is not the subjective probability of an agent but it is the subjective state probability seen by the market as a whole. This is the key idea that one must understand behind risk neutral pricing. We can price the derivative securities in the risk neutral world and not worry about individual agent's subjective probabilities. The chapter uses two methods to price a focus asset : by replication and by using state prices. The fact that these two methods yield the same result is known as *asset pricing duality.* If there is one overarching takeaway from this chapter, then it is this :

> If there is a marketed risk-free asset with total return R_f, then the risk-neutral probabilities are related to the state prices as $q = R_f\psi$. The risk neutral probabilities can be calculated from the system $R_f = R^T q$, where R is the matrix of asset returns in various states. Risk-neutral expected return on the risky assets must equal to the risk-free return.

Risk and Return in the One-Period Model

Suppose an investor is facing a choice amongst a number of investment opportunities. How does he/she go about choosing one or several of them? One of the tools that economic theory provides is the concept of utility. There are many critics to this concept. However the author says that when correct measurement units are used, all utility functions look exactly the same for small risks, and their investment advice is consistent with mean variance analysis. What is a utility function ? It is a function that satisfies certain behavioral premises for investor behavior

- Investors prefer more wealth to less
- Positive deviations from average wealth cannot compensate for equally large and equally probable negative deviations
- Risky distributions of wealth is valued by its certainty equivalent.

When the end of period wealth is uncertain, and all outcomes can be assigned a probability, one can ask what amount of certain wealth has the same utility as the expected utility of the unknown outcomes. In other words, solve

$$U(W_c) = E[U(W)]$$

The quantity of wealth W_c that solves this equation is called the certainty equivalent wealth. One is therefore indifferent between the average of the utilities of the random outcomes and the guaranteed amount W_c. If you take the utilities for all the possible values of wealth, find its average value and then map it back to the wealth that generates this average utility, one calls the obtained wealth , *certainty equivalent*. The chapter introduces three types of utility functions, CARA(constant absolute risk aversion), CRRA(constant relative risk aversion), and HARA(Hyperbolic absolute risk aversion). Out of the three classes of utility functions, HARA is the most versatile one. Obvious question is how does one use the utility functions? The usage of let's say CRRA for evaluating investment opportunities for an investor presupposes that you already know the coefficient of relative risk aversion. The chapter shows a simple example where one knows the coefficient of relative risk aversion and the investor needs to decide the optimal allocation in a risky investment. The approach one takes is as follows : Assume that some proportion of wealth is invested between risk free asset and risky asset, evaluate the utility function and compute the expected value of the utility function. Somehow I was not motivated to read this chapter as I think whole concept of utility is too academic and I don't see it using in my working life. or may I will go ever whenever there is a need. I think the only takeaway for me in this chapter is the concept of certainty equivalent.

Numerical Techniques for Optimal Portfolio Selection in Incomplete Markets

This chapter is a crash course on optimization methods to solve problems involving utility functions. Whatever utility function that one ends up using(quadratic utility/HARA/CARRA), one ends up solving an optimization problem. The basic idea is simple to grasp - Given an range of investment opportunities and the respective probabilities, you put it through an utility function and find the allocation that maximizes the expected value of the utility function. To maximize there are various numerical methods; this chapter explains newton method and gives matlab code to play with various problems.

Pricing in Dynamically complete markets

This chapter is an important one as it illustrates the techniques which are instrumental in tackling asset pricing in a dynamic framework. It starts off with building a tree for call option pricing. The example used is a 3 step tree and the first thing that needs to be done is to calibrate the up and down movement based on the stock dynamics in the real world. Calibrating means coming up with the size of the moves so that stock's average return and volatility match with the metrics that are seen in the real world. Static hedge is not enough. Why ? Wear the linear algebra lens and it becomes extremely obvious. If you were to create a static hedge that replicates option pay off, you need to find the solution to the following equation

$$\mathbf{r}_f x_1 + \mathbf{S}_T x_2 = \mathbf{V}_T$$

where r_f is a risk free rate vector, S_T is the possible stock outcomes, and V_T is the option payoff for the possible stock outcomes. RHS is non linear in stock price values and hence can never be replicated by a combination of riskfree asset and stock position at time t_0. One of the greatest insights of modern finance is that dynamic hedging can significantly reduce, and sometimes completely eliminate, the hedging error of a static hedge. It is important to realize that if one looks only one period ahead, then each node in the decision tree represents the familiar one-period model with two states (high stock return, low stock return) and two securities (stock and bond). At each node you can solve the equation $\mathbf{r}_f x_1 + \mathbf{S}_T x_2 = \mathbf{V}_T$. Linear algebra to your rescue again. This allows the hedge to be chosen separately in each node of the decision tree. Thus for the three step tree model, one can compute the option price based on iterated conditioning. The following expression

$$C_0 = E^Q\left[\frac{1}{R_f}E_1^Q\left[\frac{1}{R_f}E_2^Q\left[\frac{C_3}{R_f}\right]\right]\right]$$

becomes

$$C_0 = \frac{1}{R_f^3}E^Q\left[C_3\right]$$

The takeaway from this chapter is that you need a dynamic replication strategy to hedge an exposure to an option. It is made abundantly clear that risk neutral probabilities are a clever mathematical shortcut for writing down the no-arbitrage value of a replicating portfolio. If someone were to ask you the reason for why a static hedge is useless in option pricing, this chapter enables you to give a very sound argument that any layman can understand.

Toward Continuous time

This chapter goes through great pains to derive the risk neutral distribution of a stock at maturity. For once I felt that this book does not do justice to the amazing continuous world finance that one reads in Shreve or Baxter or other such books. You can gain some sort of intuition to continuous time finance by approaching from a discrete world view. But ultimately to solidify knowledge, one needs to slog through SDEs. I don't think any other route to continuous time finance can do justice to it. Having said that, the author deserves credit for explaining Poisson jump process in a simple way that will give confidence to any reader to access advanced texts that deal with jump diffusion processes.

Fast Fourier Transform

This chapter gives the quickest introduction to DFT that I have ever come across. To motivate the reader who has no idea of FFT, in to reading this chapter, the author gently prods by saying,

> The Fourier transform is very much about *evenly spaced points on a circle* and if you have seen a bicycle wheel, you are perfectly qualified to study this topic

There are several ways to have a visual image of convolution. The author uses a set of two concentric circles with equal number of spokes to explain convolution. I have never seen this type of visual before for describing convolution operator. I think this will go in the set of visuals that I already have in my mind for thinking about convolution. After giving the intuition and the definition of discrete Fourier transform , inverse discrete Fourier transform and convolution, the chapter explains the use of Fourier transforms in finance. If you look carefully at the various nodes of a binomial tree, there is a certain symmetry with which risk neutral probabilities are multiplied with the option prices of the subsequent node to obtain the value at the current node. One can neatly represent this pattern via DFT. By employing a variety of images, the generic formula for option pricing via discrete Fourier transform is obtained

> Consider a model with IID stock returns and a constant interest rate, represented by a recombining binomial tree with T period and $T+1$ trading dates. Le the $T+1$ dimensional vector C_T be the pay-off of the option at expiry. Let b contain the one-step state prices as the first two entries, with the remaining $T-1$ entries being zeros. Then the first element of the $T+1$-dimensional vector C_0,
>
> $$C_0 = \mathcal{F}(\mathcal{F}^{-1}(C_T) \times (\sqrt{T+1}\mathcal{F}(b))^T)$$
>
> is the no-arbitrage price of the option at time 0.

DFT is an $\mathcal{O}(n^2)$ algo. The chapter explains an algo that almost every practitioner breathes, FFT, an $\mathcal{O}(n \log n)$ algo. The algo uses a *divide and conquer* method that drastically reduce the computational speed. FFT is something that comes up in a ton of areas. Every statistician's toolkit contains density estimation techniques. Density estimation involves multiplying a set of points from a kernel function with indicator function. Any multiplication of two sequences if done repeatedly is a textbook case for FFT application. Every software that computes a density estimation using some kernel function involves FFT in its internals. The chapter concludes by giving applications of FFTs in finance. What the FFT is to practical applications, the *continuous Fourier transforms* is to theoretical work. Characteristic function $\phi(_X(\lambda))$, the Fourier transformation of the random variable X has tons of uses in stats. One can use FFT to show that the log returns are distributed normally in the limit under the risk-neutral measure. Peter Carr and Madan have written many papers that use FFT for option pricing. The key idea behind the papers is to discretize the characteristic function and cull out risk neutral probabilities numerically.

Affine processes are everywhere in finance. If you have read Shreve's volume II, they are covered at the very end of the book, which is little unfortunate as a typical course on stochastic calculus would barely able to touch this chapter in a semester. However as soon as you start exploring various stochastic processes, you realize that the math around Affine processes is a valuable skill for any quant. The way to recognize an affine process is by the specific form of its characteristic function. Affine models offer both flexibility and tractability. They permit asset returns to be serially correlated, allow for correlations across assets and also permit the presence of jumps, which is particularly important for the modeling of fat tails in asset returns. At the same time, the affine structure of the characteristic function often allows coefficients involved in the characteristic function to be evaluated in closed form which can then be used to recover risk-neutral distribution. I found this chapter to be one of the most interesting chapters in the book.

Information Management

In a one step model, the simplicity of the structure arises from the fact that the tree is a recombining tree. When is a tree recombining ? If there is a 30 step tree, there will be 2^{30} nodes. How do you handle the explosion of nodes. Markov process is the nirvana for all such problems. Under risk neutrality, the probability of achieving a certain price level depends not on the history of prices but only the previous price level. This feature of the price process makes the number of nodes well with in the control. Having said that, if the payoff is path dependent, the computations increase as you need to track each price path. Thus two factors decide the path-independent options, one is the payoff form and the other is existence of Markov process.This chapter also introduces the trick of creating conditional expectation as a random variable. Law of iterated conditioning is used in this chapter to illustrate the ease with which the pricing can be done under risk neutral measure. There are many ways to understand law of iterated conditioning. Somehow I still remember the words of my probability professor who used repeat the phrase, “smallest sigma algebra always wins”, at least once in every class. Years later I looked at it from a linear algebra perspective and the whole issue was bloody obvious. Conditional expectation is a projection operator and if you have an iterated conditioning, the expectation is dependent on that linear subspace which has the least dimension. Extremely obvious when you wear a linear algebra lens. The takeaway from this chapter for a reader is to always keep track of *state variable* as they capture all the relevant information in a given problem.

Martingales and Change of Measure in Finance

This is a short but a very important chapter as it shows the fundamental equation of asset pricing : The discounted asset price is a martingale under risk neutral measure. The martingale is created by adding shocks of zero conditional mean. Using the fact the security process is a martingale under Q, one can show that wealth of a self financing portfolio is also a martingale under Q. The chapter subsequently delves in to arbitrage theorem that links no-arbitrage condition and existence of a martingale.

Change of measure is defined as the ratio of risk neutral probability and objective probability. The intuition behind it is explained in a very beautiful way :

> If the change of measure is high in a particular scenario, then either the wealth in this scenario is very expensive or this scenario is highly unlikely to occur; in both cases it means one will not want to buy too much wealth for that scenario. Since one is buying the wealth ex ante, before the state of the market is revealed, the wealth acts as an insurance against poverty in that state. We can therefore think of states with a high change of measure as uninsurable. Conversely, states with a low change of measure are relatively cheap to insure against.

The logical progression from a discrete change of measure is a stochastic process that captures the change of measure - The famous Radon Nikodym derivative is introduced and a few examples are given to concretize the intuition. The final section of the book is interesting as it connects optimal portfolio allocation decision to change of measure. Let’s say you are an investor who is faced with certain investment opportunities that carry a certain level of risk and return. This is a multiperiod set up and hence the problem is to find out the optimal allocation $\theta_0, \theta_1, \ldots$. The author uses the fact that the investor is seeking an replicating portfolio and then sets up the problem as a convex optimization problem. The objective function of the problem is the expected utility derived from the various payoffs at a certain time T subject to the fact that the portfolio is replicating. Lagrange multiplier method is used to compute the optimal allocation. In the process, a very special relationship is uncovered

$$U'(V_T) = \lambda \frac{m_T}{\beta_T}$$

where U is the utility function, m_T is the ratio of the unconditional risk-neutral probability to the unconditional objective probability on each path in the information tree and β_t is the discounted factor. The ratio $\frac{m_T}{\beta_T}$ is called the stochastic discount factor, the pricing kernel or the state-price density. The chapter ends with a discussion on *Hansen-Jagganathan duality formula* , which links the maximum market Sharpe ratio to the Sharpe ratio of the pricing kernel

Brownian Motion and Ito Formula

This chapter deals with the definition of Brownian motion and gives the intuition behind Ito's integral. The highlight of chapter is the explanation of Martingale representation theorem(MRT) in plain simple words. If you take a discrete model and try to hedge the derivative security payoff at all the steps, there is bound to be hedging error. By moving in to continuous world, MRT guarantees a perfect replication of the derivative payoff. Usually MRT is preceded by some heavy math in many books. This book is refreshing as it presents a view where MRT's USP is presented clearly to the reader, i.e. *market becomes complete when continuous rebalancing is permitted*. The chapter does the job of introducing the famous Ito's lemma, martingales, way to identify martingales etc. One important point that any math fin student should remember is the martingale characterization of Brownian motion. This says that if both processes X_t and $X_t^2 - t$ are martingales, then X_t is a Brownian motion. The treatment of most of the concepts touched upon in the chapter is very concise and a reader who is never exposed to such concepts might find going through other books such as Shreve or Baxter, a better alternative.

Continuous-Time Finance

This chapter derives Black Scholes by various methods. First approach is via risk neutral pricing. The second approach is via Feynman-Kac formula. The third approach is via Montecarlo methods (in cases where the derivative security has state variables that are path dependent). Risk neutral pricing for a path independent option is straightforward to understand as it entails finding the conditional expectation of the discounted value of the derivative security under risk neutral measure. If the payoff is path dependent and the underlying has time varying volatilities, then one of the ways to solve the problem is to write out an SDE for the evolution of option, use the SDE's of the various processes that arise in the functional form of the option and equate the drift term to zero to get a PDE. One can then formulate boundary conditions and solve the PDE to get the prices over a multidimensional grid. Which approach should be chosen, pricing via PDEs / pricing with SDEs ? The author shows a few examples where pricing with PDEs look more like an art than science whereas pricing with SDEs appears to be relatively easy option. Also somewhere tucked in between these approaches, the author explains the market price of risk equations that enable a quant to use the risk-neutral pricing in the first place.

Finite-Difference Methods

There are entire books written on this topic. One can easily spend 3 to 4 months on getting to know intricate details of each of the finite difference method. Like everything else in this book, the author provides a solid intuition in to the various methods, gives just enough math to appreciate various methods. Basically one can start coding right away after reading this chapter. So, in that sense, the chapter is useful for someone who is just getting started on finite difference methods to solve PDEs.

Takeaway

This book is definitely a class apart from the usual books on math finance. It uses a practitioner's viewpoint to explain many concepts that are tricky to understand at the first go. The use of matrix computations, Fourier transforms, optimization methods, etc. makes this book more appealing to a practitioner rather than a theoretician. Great reference for someone working as a desk quant.