Optimal Liquidation

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Abstract

The purpose of this document is to summarize the main points of the paper "Optimal Liquidation", and provide some `R` code that gives the intuition behind Efficient Trading Frontier.

Context

For any sell side trader, optimal liquidation is his bread and butter. Given any client order, the trader has to compete against two forces, market impact and timing risk(the paper calls it volatility risk).

- Market impact : If a large trade is executed too rapidly, costs will be incurred as the trades move the market in an adverse situation
- Volatility risk : On the other hand, if the trade is executed too slowly, the the position is subject to risk during the time that the shares remain in the portfolio.

These quantities must be played off against each other by taking in to account the desired performance characteristics of the various participants. I came to know about this paper, years ago, at Courant, by Lee Maclin who works at Pragma. This is a classic paper on execution algorithms. Almgren and Chriss borrow the notion of efficient portfolio frontier and create "Efficient trading frontier" that helps in effective algo execution. At a very high level, the problem is set as a multi constraint optimization problem. A trader minimizes the cost of execution if he minimizes market impact and guards against market volatility. Hence the paper derives Pareto optimal execution paths. These paths give the trader an execution schedule for the order which he can probably feed in to a DMA and automate it. In this document, I will highlight some main points from the paper.

The basic idea of the paper :

> The cost of trading — the difference between the initial market value and the value realized after liquidation — is a random variable, whose mean and variance at the initial time depend on the liquidation strategy to be followed. This observation allows us to introduce an efficient frontier of trading strategies and define the concept of risk/reward trade off for trading strategies : reward is low transaction costs, and risk is the level of transaction costs.

Trading Model

The following are the components of the trading model :

- **Trading strategy** : X is the shares of single stock , T is the time by which the asset needs to be liquidated, τ is the discretized time step,i.e. $\tau = \kappa\tau$. Trading strategy is a list $\{x_0, \ldots, x_N\}$ where x_k

is the number of shares we hold at time t_k. The initial holding is $x_0 = X$ and the liquidation at time T requires $x_N = 0$. The trade list can also be parameterized based on number of shares that we sell between times t_{k-1} and t_k.

$$x_k = X - \sum_{j=1}^{k} n_j - \sum_{j=k+1}^{N} x_j, \quad k = 0, \ldots, N$$

The instantaneous rate of trading, in shares per minute is denoted as ν_k

$$\nu_k = \frac{n_k}{\tau} = \frac{x_{k-1} - x_k}{\tau}, \quad k = 1, \ldots, N$$

The trading strategy is deterministic. This means that given our assumptions of market impact and price volatility, the strategy gives the entire list all at once, instead of giving a partial list and then giving the full list based on some adaptive criteria.

- **Model for stock price movements** : The authors propose a trading model where the price moves via Arithmetic Brownian motion and a permanent market impact component . The authors clarify that the difference between GBM and ABM is negligible over short term horizons.

$$\begin{aligned} S_k &= S_{k-1} + \sigma\sqrt{\tau}\xi_k + \mu\tau - \tau g(\nu_k) \\ &= S_0 + \sigma \sum_{j=1}^{k} \sqrt{\tau}\xi_j + \mu t_k - \sum_{j=1}^{k} \tau g(\nu_j) \end{aligned}$$

where $g(\nu_k)$ is a function of trading ν. This term captures the trading style of the trader as it is a function dependent on the rate of trading. One must note that the impact component in the model is the permanent impact that refers to a shift in the equilibrium price of a stock under consideration due to trading. There is no temporary impact component in the equilibrium price evolution as such an impact is one-time cost and is expected to die off.

- **Permanent market impact** : A simple linear form is assumed for the permanent market impact

$$g(\nu) = \gamma\nu$$

where γ is a constant. The basic assumption here is that participants will bid low or high in proportion to the average rate of trading. Obviously one can hypothesize many types of functionals like quadratic, polynomial or whatever one thinks is the right permanent market impact function. However to keep things simple and analytically tractable, the authors use a linear model.

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \sqrt{\tau}\xi_j + \mu t_k - \gamma(X - x_k)$$

- **Temporary market impact** : A simple linear form is assumed for the temporary market impact. There is a temporary drop in price per share caused by trading at rate ν. Hence the actual price received on the sale between t_{k-1} and t_k is

$$\tilde{S}_k = S_{k-1} - h(\nu_k)$$

A linear model is considered for temporary market impact function

$$h(\nu) = \epsilon + \eta\nu$$

Thus the actual average price received on the sale between $tk - 1$ and t_k is

$$\tilde{S}_k = S_{k-1} - \epsilon - \nu v_k = S_0 + \sigma \sum_{j=1}^{k} \sqrt{\tau}\xi_j + \mu t_k - \gamma(X - x_k) - \epsilon - \eta\nu_k$$

- **Capture of the strategy** : The value received upon liquidating the entire sell program

$$X\overline{S} = SX_0 + \sigma \sum_{k=1}^{N} \sqrt{\tau} x_k \xi_k + \mu \sum_{k=1}^{N} \tau x_k - \gamma \sum_{k=1}^{N} \tau x_k \nu_k - \epsilon X - \eta \sum_{k=1}^{N} \tau \nu_k^2$$

Thus the cost of trading X units via implementation shortfall is

$$\text{Cost}(x_1, x_2, \ldots x_k) = \sigma \sum_{k=1}^{N} \sqrt{\tau} x_k \xi_k + \mu \sum_{k=1}^{N} \tau x_k - \frac{1}{2}\gamma X^2 - \epsilon X - (\eta - \frac{1}{2}\gamma\tau) \sum_{k=1}^{N} \tau \nu_k^2$$

The above cost is a random variable in $\mathbb{R}^N$. Hence one can compute the expected value and variance

$$E(x) = -\mu \sum_{k=1}^{N} \tau x_k + \frac{1}{2}\gamma X^2 + \epsilon X + (\eta - \frac{1}{2}\gamma\tau) \sum_{k=1}^{N}$$
$$V(x) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2$$

So, one can choose a specific trading schedule and get some intuitive idea about the first and second central moments of the execution cost.

The Efficient Frontier and Optimal Trading

There is an associated expected cost and level of uncertainty for any selected strategy. This means that this a multicriteria optimization problem and hence one cannot talk about single global optimum. There are a set of Pareto optimal solutions that can be generated for a specific value of risk aversion λ. The authors borrow the idea of efficient frontier for portfolio allocation and apply it to the following problem

$$\min_x E(x) + \lambda V(x)$$

The above function is convex for a given λ and hence an optimal portfolio can be found for every λ. In the words of Stanford professor, Stephen Boyd, “you can use λ as a joystick to move around the Pareto optimal frontier”.

The authors given an explicit characterization of the trading schedule for this simple problem so as to gain some intuition in to the strategy. To solve the above optimization problem, one can take the partial derivatives of the cost function with respect to each of x_k and equate it to zero and solve the resulting equations. In this case, the procedure results in a difference equation

$$\frac{1}{\tau^2}(x_{k-1} - 2x_k + x_{k+1}) = \tilde{\kappa}^2(x_k - \overline{x})$$

with

$$\tilde{\kappa}^2 = \frac{\lambda\sigma^2}{\eta(1-\gamma\tau/2\eta)}, \quad \overline{x} = \frac{\mu}{2\lambda\sigma^2}$$

One can see that $\overline{x}$ is the solution for the difference equation when the liquidation strategy is time invariant. Now how does one solve the above difference equation ? Well, the authors jump to the final solution without giving any explanation, the rationale being that it is supposed to be straightforward. Well, it is straightforward but I found it to be tedious. May be there is a better approach than the one I have taken to solve the above difference equation that makes the answer jump out quickly. My method was to solve it via Z transforms , the classic method that comes up in Linear Time Invariant Systems. I am listing a few key steps that might be helpful to someone, who is not conversant with solving difference equations.

Solving the difference equation

OK, let us solve this difference equation :

$$\frac{1}{\tau^2}(x_{k-1} - 2x_k + x_{k+1}) = \tilde{\kappa}^2(x_k - \overline{x})$$

The initial condition is $x_0 = X$ and $x_N = 0$
Take the Z transform of the equation

$$\frac{1}{\tau^2}\left(zX(z) - 2X(z) + \frac{1}{z}(X(z) - X)\right) = \tilde{\kappa}^2\left(X(z) - \frac{\overline{x}}{1-z}\right)$$

$$X(z)\left(1 - 2z\left(1 + \frac{\tilde{\kappa}^2\tau^2}{2}\right) + z^2\right) = X - \overline{x}\tilde{\kappa}^2\tau^2\frac{z}{1-z}$$

Let

$$\left(1 + \frac{\tilde{\kappa}^2\tau^2}{2}\right) = \cosh\kappa\tau$$

Thus the Z transform equation becomes

$$X(z)\left(1 - 2z\cosh\kappa\tau + z^2\right) = X - \overline{x}\tilde{\kappa}^2\tau^2\frac{z}{1-z}$$

$$X(z) = \frac{1}{(1 - 2z\cosh\kappa\tau + z^2)}X - \frac{1}{(1 - 2z\cosh\kappa\tau + z^2)}\overline{x}\tilde{\kappa}^2\tau^2\frac{z}{1-z}$$

Using partial fractions

$$X(z) = \frac{1}{(1 - 2z\cosh\kappa\tau + z^2)}X - \overline{x}\left(\frac{-1}{1-z} + \frac{-z}{1 - 2z\cosh\kappa\tau + z^2} + \frac{1}{1 - 2z\cosh\kappa\tau + z^2}\right)$$

Clubbing the relevant terms

$$X(z) = \frac{X - \overline{x}}{(1 - 2z\cosh\kappa\tau + z^2)} + \frac{\overline{x}}{1-z} + \frac{\overline{x}}{1 - 2z\cosh\kappa\tau + z^2}$$

Taking the inverse Z transform

$$x[z] = X^{-1}\left\{\frac{X-\overline{x}}{(1-2z\cosh\kappa\tau + z^2)} + \frac{\overline{x}}{1-z} + \frac{\overline{x}z}{1-2z\cosh\kappa\tau+z^2}\right\}$$

To get $x[z]$, one needs to know the inverse Z transform of $\frac{1}{(1-2z\cosh\kappa\tau+z^2)}$ and $\frac{z}{(1-2z\cosh\kappa\tau+z^2)}$.
This would involve a few more computational steps. Just for sake of completion, I will list down the steps to derive the inverse Z transform of the intermediate steps.

First let's tackle

$$X^{-1}\left\{\frac{1}{(1-2z\cosh\kappa\tau+z^2)}\right\}$$

Let $\kappa\tau = \omega$

$$X^{-1}\left\{\frac{1}{(1-2z\cosh\omega+z^2)}\right\}$$

By using the cosh expansion , one can write as

$$X^{-1}\left\{\frac{1}{(1-ze^{\omega}+z/e^{\omega}+z^2)}\right\}$$

Using partial fractions

$$X^{-1}\left\{\frac{1}{1-ze^{\omega}}\frac{1}{1-e^{-2\omega}} - \frac{1}{1-ze^{-\omega}}\frac{1}{1-e^{2\omega}}\right\}$$

$$\frac{1}{1-e^{-2\omega}}X^{-1}\left\{\frac{1}{1-ze^{\omega}}\right\} - \frac{1}{e^{2\omega}-1}X^{-1}\left\{\frac{1}{1-ze^{-\omega}}\right\}$$

Using the fact that $X^{-1}\{(1/(1-sz)\} = s^n$, the above expression simplifies to

$$\frac{e^{\omega n}}{1-e^{-2\omega}} - \frac{e^{-\omega n}}{e^{2\omega}-1} = \frac{\sinh(\omega n+\omega)}{\sinh(\omega)} = \frac{\sinh(\kappa\tau n+\kappa\tau)}{\sinh(\kappa\tau)}$$

Next let's tackle

$$X^{-1}\left\{\frac{z}{(1-2z\cosh\kappa\tau+z^2)}\right\}$$

This can be written as

$$\frac{1}{\sinh(\kappa\tau)}X^{-1}\left\{\frac{z\sinh(\kappa\tau)}{(1-2z\cosh\kappa\tau+z^2)}\right\} = \frac{\sinh(\kappa\tau n)}{\sinh(\kappa\tau)}$$

Here I have used the standard formula $X^{-1}\{\frac{z\sinh\omega}{1-2z\cosh\omega+z^2}\} = \sinh(\omega n)$
So, using the inverse Z transforms derived above, we get

$$x[k] = \frac{\sinh(\kappa\tau+\kappa\tau n)}{\sinh(\kappa\tau)}(X-\overline{x}) + \overline{x} + \overline{x}\frac{\sinh(\kappa\tau n)}{\sinh(\kappa\tau)}$$

Applying boundary condition $x[N]=0$, we get $\kappa\tau = -\kappa T$. This implies

$$x[k] = \overline{x} + \frac{\sinh(\kappa T-\kappa t_k)}{\sinh(\kappa T)}(X-\overline{x}) - \frac{\sinh(\kappa t_k)}{\sinh(\kappa T)}\overline{x}$$

The authors compute the associated velocity of trading and conclude that the solution decreases monotonically from its initial value to zero.

So, given a choice of λ, one can compute the entire trading schedule . This means by varying λ, we can get the entire family of Pareto optimal solutions.

Numerical Examples

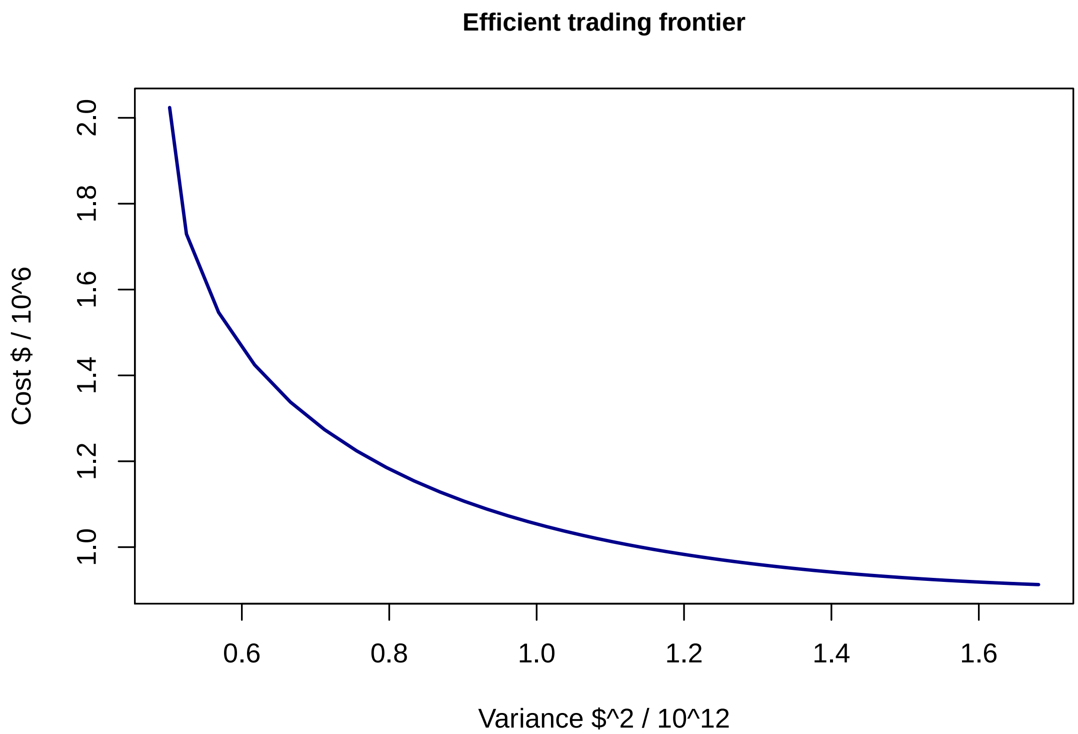
Initializing parameters

```
S0          <- 50         # Initial Stock price
X           <- 10^6       # Initial Holdings
T           <- 5          # Liquidation time
N           <- 5          # Number of time periods
tau         <- 1          # Time interval
vol         <- 0.95       # daily volatility
mu          <- 0.02        # daily growth
eps         <- 0.0625      # Bid ask spread
gamma       <- 2.5*10^(-7) # Daily volume 5 million shares
eta         <- 2.5*10^(-6) # Impact
lambda.u    <- 10^(-6)     # Static holding 11000 shares
```

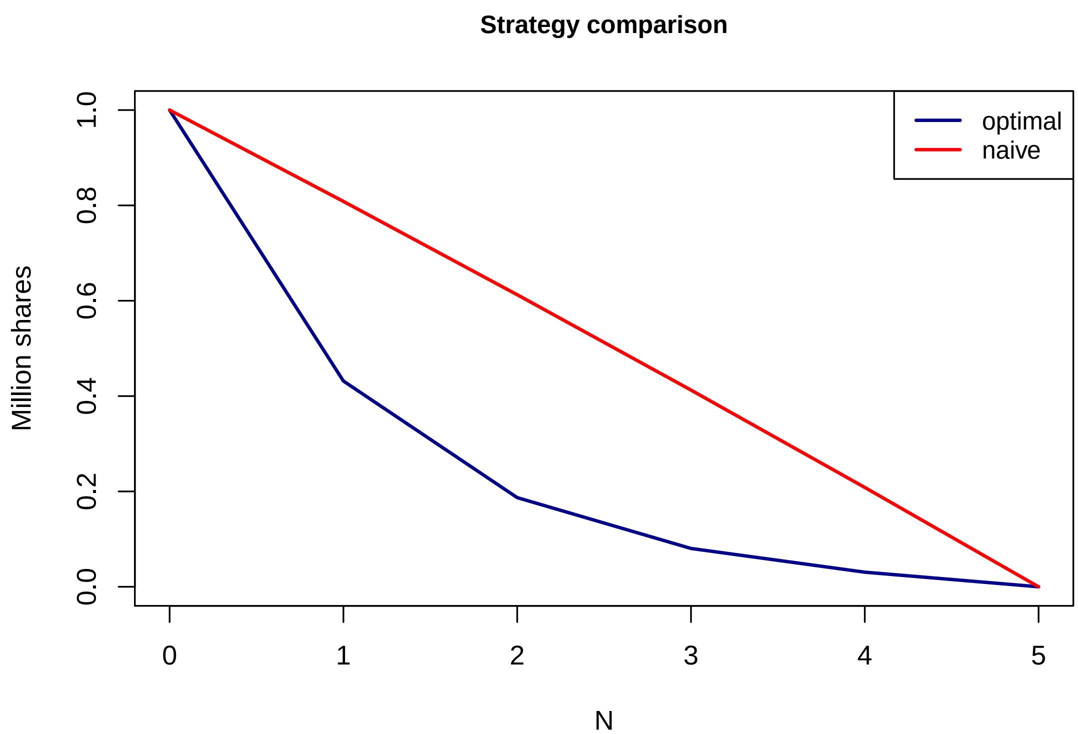
Constructing a function that gives the trading path given the parameters

```
trading.schedule <- function(lambda.u){
   kappa.tilde  <- sqrt(abs(lambda.u* vol^2 /(eta*(1-gamma*tau/(2*eta)))))
   kappa        <- acosh(kappa.tilde^2 * tau^2/2 + 1)
   tk           <- seq(0,T,tau)
   xbar         <- 0
   strategy     <- numeric(length(tk))
   if(lambda.u == 0){
   strategy     <- X*(1 - tk/T) + 0.25* mu/(eta - 0.5*gamma*tau)*tk*(T-tk)
   }else{
   xbar         <- mu / (2*lambda.u*vol^2)
   strategy     <- xbar + sinh(kappa*(T-tk))/sinh(kappa*T) *(X-xbar) -
                     sinh(kappa *tk)/sinh(kappa*T) *xbar
   }
   velocity    <- abs(diff(strategy))/tau
   exp.cost    <- - mu*sum(tau*strategy) + 1/2*gamma*X^2 + eps*X +
                    (eta-0.5*gamma*T)*sum(tau*velocity^2)
   var.cost    <- vol^2*(sum(tau*strategy^2))
   result      <- list(strategy  =strategy, cost = exp.cost, risk = var.cost)
   return(result)
}
```

```r
lambdas <- seq(0,2*10^(-5),length.out = 100 )
exp.costs <- sapply(lambdas, function(z){trading.schedule(z)$cost})
exp.var <- sapply(lambdas, function(z){trading.schedule(z)$risk})
plot(exp.costs/10^6,exp.var/10^12, xlab = "Variance $^2 / 10^12",
     ylab = "Cost $ / 10^6",
     type="l" , col = "darkblue", lwd = 2,
     main = "Efficient trading frontier", cex.main = 0.9)
```




```r
sched    <- sapply(trading.schedule(2*10^(-6)),round)$strategy /10^6
plot(0:5,sched,type="l", col = "darkblue",lwd=2, ylab = "Million shares",
     xlab = "N", main = "Strategy comparison",cex.main = 0.9)
naive    <-sapply(trading.schedule(0),round)$strategy /10^6
points(0:5,naive, type="l", col = "red",lwd=2)
legend("topright", legend=c("optimal","naive"), col=c("darkblue","red"),
       lty=1,lwd=2,cex=0.9)
```



The authors also use the properties of the Pareto optimal solutions of a convex optimization problem to show that a naive strategy ($\lambda = 0$) should never be followed. The equation used is

$$E - E_0 \approx \frac{1}{2}(V - V_0)^2 \frac{\partial^2 E}{\partial^2 V}|_{V=V_0}$$

The authors then make the point about choosing among the various efficient strategies the one to execute. This amounts to finding a way to convert a dollar of expected transaction cost into a unit of variance and vice-versa. The paper shows two ways to do it - one via utility function approach and second via VaR.The effect of increasing the parameter of temporary cost function and time to liquidation are also explored.Pretty obvious inferences here, i.e. if the market impact parameter increases, i.e. ETF shifts to the right guiding the trader to liquidate the portfolio quickly. As the time for liquidation increases, the optimal strategy shifts towards complete liquidation in the first period.The authors conclude the paper by considering multiple-stock portfolios.

Takeaway

This paper introduces the idea of "Efficient trading frontier", a framework for optimal liquidation of portfolios. Portfolio managers always have some kind of efficient frontier thinking behind their strategies. In one sense, such a Markowitz frontier was missing for the "sell side". This paper fills that void as the authors explain a framework that can serve as a rough cut quant model to start with. Obviously there are a ton of tweaks that one needs to do in the model to make it a working and practical model. But one must begin somewhere.