Laplace Transforms

• Laplace Transformation

$$f(s) = \int_0^\infty F(t)\; e^{-st} dt$$

• Given a function, its Laplace Transformation is unique

• The restrictions on the function to have a Laplace transformation are $\mid F(x) \mid \leq Me^{\alpha x}$, function should be non singular, have at most a finite number of finite jumps.

• Linearity Property

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\{F_2(t)\}$$

• First Shift Theorem

$$\mathcal{L}\{e^{-bt}F(t)\} = f(s+b)$$

• Laplace Transformations for a few functions

$$\mathcal{L}\{1\} = \frac{1}{s}$$
$$\mathcal{L}\{t\} = \frac{1}{s^2}$$
$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$
$$\mathcal{L}\{cos(t)\} = \frac{s}{s^2+1}$$
$$\mathcal{L}\{sin(t)\} = \frac{1}{s^2+1}$$
$$\mathcal{L}\{tF(t)\} = -\frac{\partial}{\partial s} f(s)$$
$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{\partial^n}{\partial s^n} f(s)$$
$$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$
$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$
$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$
$$\mathcal{L}\{\omega\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$
$$\mathcal{L}\{\int_0^t F(u)du\} = \frac{f(s)}{s}$$
$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$$
$$\mathcal{L}\left\{\int_0^t \frac{\sin(t)}{t}\right\} = \frac{1}{s}\arctan\frac{1}{s}$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \ldots - F^{(n-1)}(0)$$

• Heaviside's Unit Step Function

$$\mathcal{L}\{H(t-t_0)\} = \frac{e^{-st_0}}{s}$$

• Second Shift Theorem : If $F(t)$ is a function of exponential order in t , then

$$\mathcal{L}\{H(t-t_0)F(t-t_0)\} = e^{-st_0}f(s)$$

• Inverse Laplace Transform : If $F(t)$ has the Laplace transform $f(s)$, i.e

$$\mathcal{L}\{F(t)\} = f(s)$$

then the Inverse Laplace Transform is defined by

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

• Laplace transforms are unique apart from null functions. Inverse Laplace functions are also unique apart from null functions.

• Initial Value Theorem

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sf(s)$$

• Final Value Theorem

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sf(s)$$

• Dirac -δ function

$$\delta(t) = 0\ \forall t, t \neq 0$$
$$\int_\infty^\infty h(t)\;\delta(t)\;dt = h(0)$$

for any function $h(t)$ continous in $(\infty, -\infty)$

• Laplace Transform of δ function

$$\mathcal{L}\{\delta(t)\} = 1$$

• Filtering Property

$$\mathcal{L}\{h(t)\delta(t-t_0)\} = h(t_0)$$

• Application of Filtering Property

$$\mathcal{L}\{e^{-st}f(t)\delta(t-a)\} = e^{-sa}f(a)$$

• Relationship between $\delta(t)$ and Heaviside function

$$\int_t^\infty \delta(u-u_0)\;du = H(t-u_0)$$

Informally this means that the impulse function is the derivative of the Heaviside Unit Step Function.

- Laplace transform of Dirac -δ function and its derivatives

$$\int_{\infty}^{\infty} h(t)\ \delta(t)\ dt = h(0)$$
$$\int_{\infty}^{\infty} h(t)\ \delta'(t)\ dt = -h'(0)$$
$$\int_{\infty}^{\infty} h(t)\ \delta''(t)\ dt = h''(0)$$
$$\int_{\infty}^{\infty} h(t)\ \delta^{(n)}(t)\ dt = (-1)^n h^{(n)}(0)$$

- Laplace transformation for a periodic function

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t) dt}{1 - e^{-sT}}$$

where $F(t+T) = F(t)$,i.e., a function of period T

- The convolution of two given functions $f(t)$ and $g(t)$ is written as $f * g$ and is defined by the integral

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

- If $f(t)$ and $g(t)$ are two functions of exponential order and writing $\mathcal{L}\{f\} = \overline{f}(s)$ and $\mathcal{L}\{g\} = \overline{g}(s)$ as the two Laplace transforms, then

$$\mathcal{L}^{-1}\{\overline{f}\ \overline{g}\} = f * g$$

- Whenever you want to find a convolution between two functions, find the Laplace transformation in to frequency domain, multiply the transformations and then take the inverse Laplace transformation to get the convolution in the time domain.

- $$\mathcal{L}\{\frac{1}{\sqrt{t}}\} = \sqrt{\frac{\pi}{s}}$$

- The Error function $erf(x)$ is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

- The complementary error function $erfc(x)$ is defined by

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

- Results useful in diffusion context

$$\mathcal{L}^{-1}\{e^{-k\sqrt{s}}\} = \frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$$

$$\mathcal{L}^{-1}\{\frac{e^{-k\sqrt{s}}}{s}\} = erfc\left(\frac{k}{2\sqrt{(t)}}\right)$$

- Solving ODE's using Laplace. Typically a first order or a second order differential equation can be solved using the following Laplace transformations

$$\mathcal{L}\{f'(t)\} = s\overline{f}(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\overline{f}(s) - sf(0) - f'(0)$$

For solving an ODE, take a Laplace transformation of the ODE, the equation becomes a simple equation in $\overline{f}(s)$. Once you solve for $\overline{f}(s)$, you can apply inverse transform to get the particular and complementary solutions to the ODE.

- Solving the ubiquitous second order differential equation,

$$a\frac{\partial^2 x}{\partial t^2} + b\frac{\partial x}{\partial t} + cx = f(t)$$

The standard procedure is to take Laplace transform and convert in to an equation involving $\overline{f}(s)$ and then either use convolution formula or straight forward Laplace inversion to solve for x.

Fourier Transforms

- Bessel's Inequality : If $\{e_1, e_2, \ldots, e_n, \ldots\}$ is an orthonormal basis for the linear space V, then for each $a \in V$, the series

$$\sum_{r=1}^{\infty} \mid \langle a, e_n \rangle \mid^2$$

converges. In addition, the inequality

$$\sum_{r=1}^{\infty} \mid \langle a, e_n \rangle \mid^2 \leq \|a\|^2$$

- Fourier Series Representation of $f(x)$ that has a period 2π

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \quad -\pi < x < \pi$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

- Fourier Series Representation of $f(x)$ that has a period $2l$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l)) \quad -l < x < l$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(n\pi x/l)dx$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(n\pi x/l)dx$$

• Odd functions are represented by sines and even functions are represented by cosines

• Complex Fourier series representation

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2}(a_n - ib_n),\ c_{-n} = \frac{1}{2}(a_n + ib_n),\ c_0 = \frac{1}{2}a_0$$
$$c_n = \frac{1}{2\pi}\int_{\pi}^{\pi} f(x)e^{-inx} \text{ and } c_{-n} = \frac{1}{2\pi}\int_{\pi}^{\pi} f(x)e^{inx}$$

• Properties of Fourier Series
If f(x) is represented by the following Fourier Series

$$f'(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \quad -\pi < x < \pi$$

then

$$f'(x) \sim \sum_{n=1}^{\infty}(-na_n\cos(nx) + nb_n\sin(nx)) \quad -\pi < x < \pi$$

and

$$\int_{-\pi}^{x} f(t)dt = \frac{1}{2}a_0(x+\pi) + \sum_{n=1}^{\infty}(\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}(\cos(nx) - \cos(n\pi))) \quad -\pi < x < \pi$$

and the function on the right converges uniformly to the function on the left.

• A fourier series expansion of $f(x)$ can be point wise convergent or uniformly convergent. If it is uniformly convergent, then you can differentiate term by term. If it is pointwise convergent, then differentiating both sides of the equation gives nonsense results

• If $f(t)$ and $g(t)$ are continous $(-\pi,\pi)$ and provided

$$\int_{\pi}^{\pi} \mid f(t) \mid^2 dt < \infty and \int_{\pi}^{\pi} \mid g(t) \mid^2 dt < \infty$$

if a_n and b_n are the Fourier coefficients of $f(t)$ and α_n, β_n those of $g(t)$, then

$$\int_{-\pi}^{\pi} f(t)g(t)dt = \frac{1}{2}\pi a_0 b_0 + \pi\sum_{n=1}^{\infty}(\alpha_n a_n + \beta_n b_n)$$

• Parseval Identity :If $f(t)$ is continous in the range$(-\pi,\pi)$, is square integrable and has Fourier coefficients a_n, b_n, then

$$\int_{-\pi}^{\pi}[f(t)]^2 dt = 2\pi a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

• Fourier Half Range series, i.e fourier series expansion over the interval $(0,\pi)$. If $f(x)$ is assumed as an even function over $(-\pi,0)$, then all b_n's are 0. If $f(x)$ is assumed as an odd function over $(-\pi,0)$, all a_n's are 0.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

• Fourier half range series are very useful in specifying the boundary condition for the heat equation. Since the boundary condition is specified for $[0,L]$, depending on whether the terms in the boundary condition are even or odd, an appropriate form of Fourier half range series can be used.

• Solving heat equation by using Separation of Variables technique where Laplace transformation can be used to solve the second order ODE ,Half range Fourier series can be used to represent the initial value condition.

• Laplace transformation can be used to solve PDEs. The symbol t, where denotes time in PDEs that ranges from $(0,\infty)$ corresponds neatly to the range of the Laplace transform.

• Use of Fourier and Laplace transform and such analytical methods have been surpassed by computers that be solve a PDE using numerical methods. However analytical method gives the intuition behind the solution that is not so obvious from the numerical solution.